Def: let X be a Riemann surface, and $f: X \circlearrowleft$ a holomorphic selfmap.
(↖ always assumed non-constant)
The Fatou set of f is given by:

$$\mathcal{F}(f) = \{x \in X \mid \exists\, U \ni x \text{ neighborhood s.t. } \{f^n|_U\} \subset \mathrm{Hol}(U, X) \text{ is normal}\},$$

i.e., the locus where the family of iterates of f is (locally) normal.

We define the Julia set of f as $\mathcal{J}(f) = X \setminus \mathcal{F}(f)$.

Rem: in the compact case, $\mathcal{F}(f)$ is the local equicontinuity locus of $\{f^n \mid n \in \mathbb{N}\}$.

Example: $f: \mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$, $\mathcal{F}(f) = \mathbb{C} \setminus \partial\mathbb{D}$, $\mathcal{J}(f) = \partial\mathbb{D}$.

$f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $z \mapsto z^2$, $\mathcal{F}(f) = \hat{\mathbb{C}} \setminus \partial\mathbb{D}$, $\mathcal{J}(f) = \partial\mathbb{D}$.

Rem: by definition, $\mathcal{F}(f)$ is an open subset of X, and $\mathcal{J}(f)$ is a closed subset.

Prop: (Invariance) $\mathcal{J}(f)$ and $\mathcal{F}(f)$ are totally invariant subsets of X.

Proof: If we prove the statement for $\mathcal{F}(f)$, the one for $\mathcal{J}(f)$ easily follows.
We want to show that $x \in \mathcal{F}(f) \iff f(x) \in \mathcal{F}(f)$.

$x \in \mathcal{F}(f) \iff \exists\, U \ni x$ such that $\{f^n|_U\}$ is normal, i.e., $\forall\, n_k$ subsequence (in theory a subset of $\mathbb{N}$, but we can always assume it is increasing by taking subsequences) $\exists\, n_{k_h}$ subsequence so that $f^{n_{k_h}}$ either converges uniformly or diverges uniformly from X.

But this happens $\iff$ $f^{n_{k_k}-1}$ does the same on $f(U)$. $\square$

~~Hence~~

Prop: $\forall m>0$, $\mathcal{J}(f^m)=\mathcal{J}(f)$ (and similarly $\mathcal{F}(f^m)=\mathcal{F}(f)$.)

Proof: Set $\mathcal{A}=\{f^n \mid n\in\mathbb{N}\}$, $B_k=\{f^{mn+k} \mid n\in\mathbb{N}\}$ $\forall k=0,\dots,m-1$.

Notice that $\mathcal{A}=\bigcup_{k=0}^{m-1} B_k$. Since $\mathcal{A}\supseteq B_0$, if $\mathcal{A}$ is normal locally at x, so is B_0, and $\mathcal{F}(f)\subseteq\mathcal{F}(f^m)$.

Suppose now that B_0 is locally normal at x.

We will show that B_1 (hence B_k $\forall k$) is also locally normal at x.

Since the union of finitely many normal families is a normal family, this concludes the proof.

In fact, for any sequence in $\mathcal{A}$, $\exists k$ such that B_k contains infinitely ~~many~~ elements of such sequence, ...

By hypothesis, $\exists U\ni x$ open so that any sequence $(f^{mn}|_U)_{n\in N_1}$ admits a subsequence $(f^{mn}|_U)_{n\in N_2}$ that either converges uniformly or diverges uniformly from ~~X~~.

Fix any $x\in\mathcal{F}(f^m)$ (where B_0 is locally normal), and fix $U\ni x$ satisfying the above property.

Consider any sequence $(f^{nm+1})_{n\in N_1}$ in B_1, and its shifted sequence $(f^{nm})_{n\in N_1}$ in A. By normality there exists a subsequence $(f^{nm})_{n\in N_2}$ that either converges uniformly or diverges uniformly from X.

Suppose we are in the first case, and $f^{nm} \underset{\substack{n\to\infty \\ n\in N_2}}{\longrightarrow} g$. Then $f^{nm+1} \underset{\substack{n\to\infty \\ n\in N_2}}{\longrightarrow} f\circ g$ uniformly.

Suppose we are in the second case, and (f^{nm}) diverges from ~~compact~~ X.

We show that $(f^{nm+1})_{n\in N_2}$ also diverges unif. from X.

$\forall K\subset U$, $L\subset X$ compact sets; $f^{-1}(L)\overset{=}{}L'$ is also compact, being f proper.

Then $\exists N = N(K,L')$ such that $\forall n>N$, $f^{nm}(K)\cap L' = \emptyset$ $(n\in N_2)$

Then $f^{nm+1}(K)\cap L = \emptyset$, and (f^{nm+1}) diverges unif. from X $\square$

Fixed / Periodic points.

Def: $f: X\to X$ holomorphic map. $z\in X$ is:

- a fixed point if $f(z)=z$ (set: $Fix(f)$).
- a periodic point if $\exists m\in\mathbb{N}^*$, $f^m(z)=z$. Any such m is a period of z.
  The minimal such m is called the (exact) period of z.

$$z_0 \overset{f}{\longmapsto} z_1 \to z_2 \to \cdots \to z_{m-1} \to z_0$$

$\leftarrow$ is called a periodic orbit (or a cycle) (of length m).

The multiplier of a periodic point / orbit is:

$$\lambda = (f^m)'(z_0) = f'(z_0)\cdot\ldots\cdot f'(z_{m-1}) \in \mathbb{C}.$$
$\quad$ (if $X=\mathbb{C}$)

- a preperiodic point if $\exists n>m\geq 0$, $f^n(z)=f^m(z)$. $\Leftrightarrow$ z has finite orbit.

~~The point z~~ A periodic point z is said: ($\lambda = f^{m'}(z)$)

- superattracting if $\lambda = 0$
- attracting if $0 < |\lambda| < 1$

} contracting (in literature sometimes $|\lambda|<1$ is called attracting)

- repelling if $|\lambda| > 1$
- indifferent (or neutral) if $|\lambda| = 1$
  - parabolic if λ is a root of unity.
  - irrational if $\lambda = e^{2\pi i t}$, $t \in \mathbb{R} \setminus \mathbb{Q}$.

usually we assume that f has not finite order, i.e. $f^n \neq \mathrm{id}$ $\forall n \in \mathbb{N}^*$.

Rem: Notice that if $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and $f^m(\infty) = \infty$, (may assume $m=1$) lor then

$f^{m'}(\infty)$ is not the limit of $f'(z)$ for $z \to \infty$, but its reciprocal

$f(z) = \dfrac{P(z)}{Q(z)}$. ∞ is fixed by $f \iff \deg P > \deg Q$.

To compute $f'(\infty)$, we consider the chart $w = \frac{1}{z}$, and in local coordinates:

$f(w) = \dfrac{Q(\frac{1}{w})}{P(\frac{1}{w})}$ If $P(z) = z^p(\alpha + o(1))$, $Q(z) = z^q(\beta + o(1))$.

Then $f(z) = z^{p-q}\left(\frac{\alpha}{\beta} + o(1)\right)$, and $f'(z) = (p-q)\, z^{p-q-1}\left(\frac{\alpha}{\beta} + o(1)\right)$

where $o(1)$ are functions that tend to 0 when $z \to \infty$ ($\frac{1}{z} = w \to 0$)

$$f(w) = \frac{w^{-q}(\beta + o(1))}{w^{-p}(\alpha + o(1))} = w^{p-q}\left(\frac{\beta}{\alpha} + o(1)\right) \to f'(w) = w^{p-q-1}\left(\frac{\beta}{\alpha} + o(1)\right)(p-q)$$

Hence if $p - q = 1 \Rightarrow f'(z) \to \frac{\alpha}{\beta}$ and $f'(\infty) = \frac{\beta}{\alpha}$.

if $p - q \geq 2 \Rightarrow f'(z) \to \infty$ and $f'(\infty) = 0$.

(4.5)

The name "contracting" comes from the following property

Prop: Let $f: X \to X$ be a holomorphic map with fixed point $z_0 = f(z_0)$

z_0 is contracting $\Rightarrow \exists U \ni z_0$ neighborhood so that $f^n(z) \to z_0 \ \forall z \in U$.

Rem: the convergence is exponential (and faster if z_0 is superattracting).

Proof. Being the property local, we may assume $X \subset \mathbb{C}$ and $z_0 = 0$.

We may write $f(z) = \lambda z(1 + u(z))$, locally at 0, where $u(0) = 0$.

Lemma: $\forall \Lambda, |\lambda| < \Lambda, \exists r > 0$ such that $|f(z)| \leq \Lambda |z|$.

Proof: $\Lambda = |\lambda| \cdot (1+\varepsilon)$ $\varepsilon > 0$, by continuity $\exists r$ s.t. $|u(z)| < \varepsilon \ \forall z, |z| < r$. $\square$

($\Rightarrow$) If $|\lambda| < 1$, we may pick $\Lambda < 1$, $|\lambda| < \Lambda$, and by the Lemma, $\exists r > 0$ s.t. $\forall z, |z| < r \Rightarrow |f(z)| \leq \Lambda |z|$. Being $\Lambda < 1$, $\Lambda |z| < r$, and we can iterate:

$\forall z, |z| < r, \quad |f^n(z)| \leq \Lambda^n |z| \xrightarrow[n\to\infty]{} 0$.

($\Leftarrow$) Assume there exists r so that $\forall z \in D_r = D(0, r)$, $f^n(z) \to 0$.

It follows that $\exists n >> 0$ such that $f^n(D_r) \subset D_{\frac{r}{2}}$. (~~Maybe~~ consider $\overline{D_r}$, compact!

For any z $\exists$ such $n = n(z)$. By continuity, the same n works in a nbhd of z. By compactness, extract a finite covering of $\overline{D_r}$, and take the greatest $n(z)$.

By Cauchy derivative estimate, $|\lambda|^n = |f^n{}'(0)| \leq \dfrac{\frac{r}{2}}{r} = \dfrac{1}{2}$, and f is contracting at 0. $\square$

Definition: Let z_0 be a contracting fixed point for f ~~...~~ $= d_f(z_0)$

The basin of attraction of z_0 is the open set $\mathcal{A} \subset X$ given by

$\mathcal{A} = \{ z \in X \mid f^n(z) \to z_0 \}$ ← open by the Lemma; $\mathcal{A} = \bigcup_{n \in \mathbb{N}} f^{-n}(U)$.

Similarly, if $O_f(z_0)$ is an contracting cycle (the orbit of a periodic point z_0), then its basin of attraction is

$\mathcal{A} = \{ z \in X \mid f^{nm}(z) \to p$ for some $p \in O \} \overset{= \bigcup_{p \in O} \mathcal{A}_{f^m}(p)}{}$, m the period of z_0.

The connected component of $\mathcal{A}_0$ the basin of attraction $\mathcal{A}$ of a fixed point z_0 is called the "immediate basin of attraction".

Proposition: $f : X \to X$ holomorphic selfmap on a Riemann surface.
Every contracting periodic (cycle) orbit (and its basin of attraction) belongs to $\mathcal{F}(f)$ the Fatou set.

· Every repelling cycle belongs to the Julia set $\mathcal{J}(f)$

Proof. Since $\mathcal{F}(f^m) = \mathcal{F}(f)$ and $\mathcal{J}(f^m) = \mathcal{J}(f)$, we may assume we have a fixed point z_0.

If z_0 is contracting, then it follows from the lemma that $\exists U \ni z_0$ nbhd s.t. Any sequence in $\{f^n|_U\}$ converges uniformly to the constant function z_0. Hence $U \subset \mathcal{F}(f)$, and since $\mathcal{F}(f)$ is totally invariant, $\mathcal{A}_f(z_0) \subset \mathcal{F}(f)$.

. If z_0 is repelling, we show that $\{f^n|_U\}$ is not normal $\forall U \ni z_0$ nbhd.
In fact, no subsequences of this family may converge uniformly on compact subsets, since $(f^n)'(z_0) = \lambda^n \to \infty$.
No subsequences may also diverge from X, since $f(z_0) = z_0$
(as $f^n(\{z_0\}) \cap \{z_0\} \neq \emptyset$ $\forall n$). $\square$

The neutral case is much more complicated.

Recall that a fixed point is parabolic if f has not finite order ($f^n = \mathrm{id}\ \forall n$) and its multiplier is a root of unity.

Prop: Every parabolic periodic point belongs to the Julia set.

Proof: Up to replacing f by an iterate, we may assume that z_0 is a fixed point, and $f'(z_0) = 1$. In local coordinates:

$$f(z) = z(1 + a_r z^r + \underset{\text{Higher order terms}}{\underbrace{h.o.t.}}) \qquad \text{where } r \in \mathbb{N}^*,\ a_r \neq 0.$$

By direct computation, $f^n(z) = z(1 + n a_r z^r + h.o.t.)$.

It follows that no sequences in $\{f^n|_U\}$ may converge uniformly locally at 0, since $n a_r \underset{n\to\infty}{\to} \infty$ (and $f^{(r+1)}(0) = n a_r (r+1)! \underset{n\to\infty}{\to} \infty$).

$f^n|_U$ cannot diverge from X either, since $0 \in U$ is a fixed point. $\square$

The irrational case ($\lambda = e^{2\pi i \theta}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$) is more complicated and an irrational fixed point may belong to either $J(f)$ or $F(f)$.

We now focus on the case of rational functions $f: \hat{\mathbb{C}} \circlearrowleft$.

Set $d = \deg f$.

$d = 0$: f is constant, and $F(f) = \hat{\mathbb{C}}$, $J(f) = \emptyset$.

$d = 1$: $f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$. f has at least a fixed point. By transitivity, may suppose it is ∞, and $f(z) = az + b$, $a \neq 0$.

If $a \neq 0$, f has another fixed point, by transitivity may assume 0, and $f(z) = az$.

- $|a| < 1$. 0 is attracting, ∞ repelling, $J(f) = \{\infty\}$ $F(f) = \mathbb{C}$.
- $|a| > 1$. analogous (apply to f^{-1}), $J(f) = \{0\}$.
- $|a| = 1$: Any sequence a^{n_k} admits a converging subsequence

It follows that $\{f^n\}$ is a normal family, $F(f)=\hat{\mathbb{C}}$, $J(f)=\emptyset$.

If $a=1$: $f(z)=z+b$ $\begin{cases} b=0 & J(f)=\emptyset. \\ b\neq 0: & \infty \text{ is a parabolic fixed point.}\end{cases}$

$\{f^n|_{\mathbb{C}}\}$ is a normal family since any sequence converges uniformly on compact sets to ∞. $\Rightarrow$ $J(f)=\{\infty\}$, $F(f)=\mathbb{C}$.

Assume $d\geq 2$ from now on.

Prop: $f: \hat{\mathbb{C}} \circlearrowleft$, $\deg f \geq 2$ $\Rightarrow$ $J(f)\neq\emptyset$.

Proof: If by contradiction $J(f)=\emptyset \rightsquigarrow F(f)=\hat{\mathbb{C}}$,

$\rightsquigarrow \exists\, n_j$ subsequence so that $f^{n_j} \to g: \hat{\mathbb{C}}\circlearrowleft$ uniformly.

We claim that $\deg(f^{n_j}) = \deg g$ for $j>>0$.

If this is true, we would get a contradiction, since $\deg(f^{n_j}) = d^{n_j}\to\infty$.

To prove the claim: set $d_g=\deg(g)$.

If $d_g=0$, then $g\equiv c$ constant. Up to change of coordinates we may assume $c=0$. Then f^{n_j} is bounded hence constant for $j>>0$, and we are done.

If $d_g\geq 1$: Up to action of $\mathrm{Aut}(\hat{\mathbb{C}})$, we may assume that

$g^{-1}(0)=\{z_1, \dots, z_{d_g}\}$ (g has all distinct zeroes, all in $\mathbb{C}$).

Take D_k, $k=1,\dots,d_g$ discs of the form $D_k=D(z_k,r)$, with $r>0$ small enough so that D_k are all distinct and contain no poles of g.

On $\bigcup \partial D_k$ $|g(z)|$ takes a minimum, say $\varepsilon>0$. By uniform convergence, for $j>>0$, $|f^{n_j}(z)-g(z)|<\varepsilon\leq|g(z)|$. By Rouché's theorem,

$\#\text{zeroes}(f^{n_j}|_{D_k}) = \#\text{zeroes}(g|_{D_k})$

On $K = \hat{\mathbb{C}} \setminus \bigcup_k D_k$, $\nexists\, g(z) \neq 0 \Rightarrow |g(z)| \geq \varepsilon$ on K. (K compact)

(by max modulus principle the max of $|\frac{1}{g}|$ is taken on the boundary ...)

Again for $j \gg 0$, $|f_a^{n_j}(z) - g(z)| < \varepsilon$, and $f^{n_j}(z) \neq 0$ $\forall z \in K$.

Hence $\underset{\deg(f^{n_j})}{\# \text{zeros}\,(f^{n_j})} = \underset{\deg g}{\# \text{zeros}\,(g)}$ $\square$

Grand orbits:

Def: The grand orbit of a point $z \in X$ under $f: X \to X$ is the set $GO_f(z) = \{ z' \in X \mid \exists\, n, m \geq 0,\ f^n(z) = f^m(z') \} =$

$= \bigcup_{m=0}^{+\infty} f^{-m}(O_f(z))$ where $O_f(z)$ is the (forward) orbit.

A point z is called exceptional if it has finite grand orbit. ← set $\mathcal{E}(f)$

Prop: $f: \hat{\mathbb{C}} \circlearrowleft$, $\deg f =: d \geq 2$. Then the set $\mathcal{E}(f)$ of exceptional points can have at most two elements.

Moreover, any $z \in \mathcal{E}(f)$ is necessarily a superattracting periodic point, hence $\mathcal{E}(f) \subset \mathcal{F}(f)$.

Proof:

Being f surjective, it sends a grand orbit $GO_f(z)$ surjectively on itself.

If $GO_f(z)$ is finite then $f|_{GO_f(z)}$ is a bijection of $GO_f(z)$.

Hence it constitutes a single cycle $z_0 \mapsto z_1 \mapsto \dots \mapsto z_{m-1} \mapsto z_m = z_0$.

In particular, $f^{-1}(z_j) = \{z_{j-1}\}$ $\forall j = 1 \dots m$. Since $\# f^{-1}(z) = d$ (counted with multiplicity) it follows that the multiplicity of f at any z_j is $d \geq 2$, and $z_0, \ldots, z_{m-1}$ is a superattracting cycle (hence $z_j \in F(f)$).

Suppose that $\# \mathcal{E}(f) \geq 3$. Set $U = \hat{\mathbb{C}} \setminus \mathcal{E}(f)$.

Since $f^{-1}(\mathcal{E}(f)) \subset \mathcal{E}(f)$, we get $f(U) \subset U$, and $f: U \to U$ defines a dynamical system. Since U is hyperbolic, $\{f^n|_U\}$ is normal, and $U \subset F(f)$. But $\mathcal{E}(f) \subset F(f)$, and we get $F(f) = \hat{\mathbb{C}}$, i.e. $J(f) = \emptyset$ a contradiction $\square$

Rem: for $X \neq \hat{\mathbb{C}}$, exceptional points need not be superattracting.

E.g.: $f(z) = \lambda z e^z$. is $\mathcal{E}(f) = \{0\}$, and $f'(0) = \lambda$.

Theorem (transitivity of $J(f)$)

Let $z_0 \in J(f) \subset \hat{\mathbb{C}}$ be any point, and U an arbitrary neighborhood of z_0. Then $\bigcup_{n \in \mathbb{N}} f^n(U) \supseteq \hat{\mathbb{C}} \setminus \mathcal{E}(f)$.

Rem: in fact $f^n(U) \supset J(f)$ for $n \gg 0$, we will see this later)

Proof: Call $\Omega = \bigcup_{n \in \mathbb{N}} f^n(U)$.

Then $\hat{\mathbb{C}} \setminus \Omega$ contains at most 2 points.

In fact, $f(\Omega) \subseteq \Omega$, hence $f: \Omega \circlearrowleft$ defines a dynamical system.
If $\hat{\mathbb{C}} \setminus \Omega$ contains at least 3 points, Ω is a hyperbolic Riemann surface and $\{f^n|_\Omega\}$ is normal, against the hypothesis $z_0 \in U \cap J(f)$.

Rem: Ω may be not connected but contained in some Ω' hyperbolic (connected) apply the normality to $\{f^n|_U : U \to \Omega'\}$.

Since $f(\Omega) \subset \Omega$, we have $f^{-1}(\hat{\mathbb{C}} \setminus \Omega) \subset \hat{\mathbb{C}} \setminus \Omega$, and $\hat{\mathbb{C}} \setminus \Omega \subset \mathcal{E}(f)$. □

Corollary. If the Julia set $J(f)$ contains an interior point, then $J(f) = \hat{\mathbb{C}}$.

Proof. Apply the theorem to $z_0 \in \mathring{J}(f)$, $U \subset J(f)$ neighborhood of z_0.

$\Rightarrow \Omega = \bigcup_{n \geq 0} f^n(U)$. contains $\hat{\mathbb{C}} \setminus \mathcal{E}(f)$.

Being $J(f)$ totally invariant, $\hat{\mathbb{C}} \setminus \mathcal{E}(f) \subset \Omega \subset J(f)$.

Being $J(f)$, closed and $\mathcal{E}(f)$ finite, we deduce $J(f) = \hat{\mathbb{C}}$. □

Corollary: If $\mathcal{A} \subset \hat{\mathbb{C}}$ is the basin of attraction of an ~~the~~ contracting cycle, then $\partial \mathcal{A} = J(f)$.

Every connected component of $F(f)$ is either a connected component of $\mathcal{A}$ or disjoint from $\mathcal{A}$.

Proof: $J(f) \subseteq \partial \mathcal{A}$:

Let $z_0 \in J(f)$ be any point in the Julia set, and $U \ni z_0$ any neighborhood.

By the theorem $\bigcup_n f^n(U) \cap \mathcal{A} \neq \emptyset$ ($\mathcal{A}$ has infinitely many points...)

$\Rightarrow U \cap \mathcal{A} \neq \emptyset$ ($\mathcal{A} = f^{-1}(\mathcal{A})$)

Hence $z \in \overline{\mathcal{A}}$. But $z \notin \mathcal{A}$, hence $z \in \partial \mathcal{A}$.

$J(f) \supseteq \partial \mathcal{A}$: Let $z_0 \in \partial \mathcal{A}$, and $V \ni z_0$ any neighborhood of z_0.

For any point $z \in V \cap \mathcal{A}$, the limit points of $\{f^n(z)\}$ are the attracting cycle in $\mathcal{A}$, that is at non zero distance from $\partial \mathcal{A}$.

$\forall z \in V \setminus \mathcal{A}$, $f^n(z) \in \hat{\mathbb{C}} \setminus \mathcal{A}$. It follows that any pointwise limit of $f^{n_j}|_V$ would be not continuous at z_0, and $z_0 \in J(f)$. □

Finally, any component of $F(f)$ which intersects $\mathcal{A}$ must be a connected component of $\mathcal{A}$, since $F(f) \cap \partial\mathcal{A} = \emptyset$. □

Rem: $\partial\mathcal{A} \underset{\text{in general}}{\supsetneq} \bigcup_{\text{conn. comp. of } \mathcal{A}} \partial U$

Example: K = cantor set is uncountable but the boundary of the union of the c.c. of the complement is countable.

Corollary: $\forall z_0 \in J(f)$, the set $\mathcal{O}_f^-(z_0) = \{ z \in \hat{\mathbb{C}} \mid f^n(z) = z_0, n \in \mathbb{N}\}$ is dense in $J(f)$: $\overline{\mathcal{O}_f^-(z_0)} = J(f)$.

Proof: Since $z_0 \in J(f)$, $J(f)$ is closed and totally invariant, we infer $\overline{\mathcal{O}_f^-(z_0)} \subseteq J(f)$.

Let now $z \in J(f)$. $\forall U \ni z$, neighborhood $\bigcup_{n \geq 0} f^n(U) \supset \hat{\mathbb{C}} \setminus \mathcal{E}(f) \ni z_0$

Hence there exists n such that $z_0 \in f^n(U)$, i.e., $\exists w \in U, f^n(w) = z_0$,

i.e. $w \in U \cap \mathcal{O}_f^-(z_0)$ and $z \in \overline{\mathcal{O}_f^-(z_0)}$. □

Rem: To draw $J(f)$ one can use these theorems:

- If we know that f has some contracting cycle we may draw its basin of attraction, and its boundary is the Julia set.
This happens for example for polynomials (with superattracting fixed point at ∞) or the Newton method (with contracting fixed points the zeroes of the associated polynomial).

- In general one can start from any $z_0 \in J(f)$, and compute its backward orbit.
We will see that f tends to be expanding around $J(f)$ (all repelling cycles are there), hence f^{-1} tends to be contracting.

These results have measure-teoretic counterparts, we may see them later

Corollary: $f: \hat{\mathbb{C}} \circlearrowleft$ of degree $d \geq 2$. $\Rightarrow$ $J(f)$ is infinite, with no isolated points (is perfect)

Proof: If by contradiction $J(f)$ is finite, being totally invariant, it would be $J(f) \subseteq \mathcal{E}(f)$, which is absurd ($\mathcal{E}(f) \subset F(f)$)

Being $\hat{\mathbb{C}}$ compact, $J(f)$ admits at least an accumulation point $z_0 \in J(f)$.

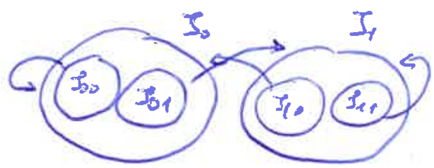
Then $O_f^-(z_0)$ forms a dense subset of $J(f)$ of non-isolated points $\square$

Theorem: $f: \hat{\mathbb{C}} \circlearrowleft$ of degree $d \geq 2$. Then $J(f)$ is either connected, or it has uncountably many connected components.

Proof: Suppose $J(f)$ is not connected. Then $J(f) = J_0 \sqcup J_1$ is the disjoint union of two compact ~~sets~~ non-empty sets J_0, J_1.

Since $J(f)$ ~~...~~ has no isolated points, so is for J_0 and J_1, which are infinite.

For any $z \in J(f)$, we denote by $\beta_n(z) \in \{0,1\}$ the index so that $f^n(z) \in J_{\beta_n(z)}$ and denote:



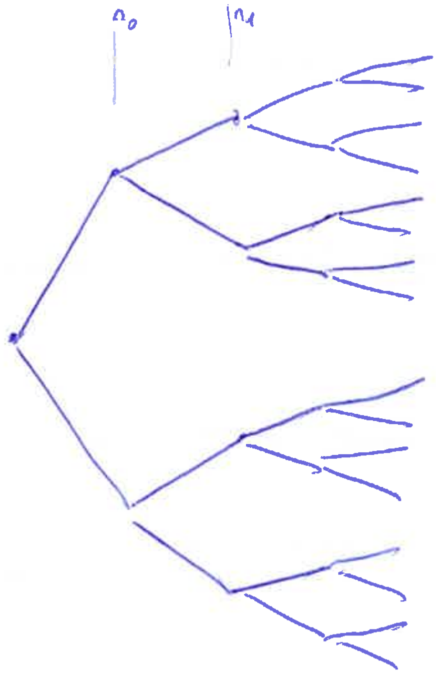
$\beta_{\mathbb{N}}(z) = (\beta_n(z), n \in \mathbb{N})$, and $\beta_{\leq k}(z) = (\beta_n(z) \mid n \leq k)$.

$J_{\alpha_0 \dots \alpha_k} = \beta_{\leq k}^{-1}(\alpha_0, \dots, \alpha_k)$.

Notice that if z, z' belong to the same connected component, then $\beta_{\mathbb{N}}(z) = \beta_{\mathbb{N}}(z')$.

Lemma: $\forall z \in J(f)$, $\forall k \in \mathbb{N}$ $\exists z' \in J(f)$ so that $\beta_{\leq k}(z) = \beta_{\leq k}(z')$ but $\beta_{\mathbb{N}}(z) \neq \beta_{\mathbb{N}}(z')$.

If the lemma holds, then from any $z \in J(f)$ we may build a tree as follows:



$\beta_{\mathbb{N}}(z)$: ~~given~~ follows:

given a z, pick $k = 0$, and there exists z' s.t. $\beta_{\mathbb{N}}(z) \neq \beta_{\mathbb{N}}(z')$ but $\beta_0(z) = \beta_0(z')$.

$\Rightarrow \beta_{n_0}(z) \neq \beta_{n_0}(z')$

Now pick $k \geq n_0$

apply the lemma to z, z', there are other two sequences realised, all different for some level n_1. Apply the lemma for $k = n_1$, and so on.

← this set has cardinality $2^{\mathbb{N}}$, hence uncountable.

Proof of lemma.

Set $J_{\alpha_0 \ldots \alpha_k} = \beta_{\leq k}^{-1}(\alpha_0 \ldots \alpha_k) = \{ z \in J(f) \mid f^n(z) \in J_{\alpha_n} \ \forall n = 0 \ldots k \}$.

set $U_\alpha = \hat{\mathbb{C}} \setminus J_{1-\alpha} = F(f) \cup J_\alpha$, and $U_{\alpha_0 \ldots \alpha_k} = \{ z \in \hat{\mathbb{C}} \mid f^n(z) \in U_{\alpha_n} \ \forall n = 0 \ldots k \}$

$= F(f) \cup J_{\alpha_0 \ldots \alpha_k}$.

Assume that there exists $\alpha_0 \ldots \alpha_k$ so that $\beta_{\mathbb{N}}(z) = \beta_{\mathbb{N}}(z') \ \forall z, z' \in J_{\alpha_0 \ldots \alpha_k}$.

$\alpha_{\mathbb{N}} = (\alpha_n)$

The sequence of α_n must contain infinitely many 0's or 1's (or both). Up to relabelling, may assume 0's.

for any infinite sequence of iterates

It follows that there exists a subsequence n_j so that $f^{n_j}(J_{\alpha_0 \ldots \alpha_k}) \subset J_0 \ \forall j$.

and hence $f^{n_j}(U_{\alpha_0 \ldots \alpha_k}) \subset U_0$.

But $U_0 = \hat{\mathbb{C}} \setminus J_1$ is hyperbolic, hence the family $\{ f^{n_j}|_{U_{\alpha_0 \ldots \alpha_k}} \}$ is normal and there exists a subsequence converging. This implies that $U_{\alpha_0 \ldots \alpha_k} \subset F(f)$, which is a contradiction since $z \in J_{\alpha_0 \ldots \alpha_k} \subset U_{\alpha_0 \ldots \alpha_k}$. □

Rem: ~~The~~ The construction in the previous proof resembles the one of Cantor sets: the difference is that for Cantor sets $I_{a_0 \ldots a_n} \neq \emptyset$ $\forall a_0 \ldots a_n$, here we have this property for enough (a_i), so we have still uncountably many components

~~Baire~~ Baire spaces:

Def: A topological space X is a Baire space if every countable intersection of dense open subsets of X is dense.

Baire's theorem: every complete metric space is a Baire space (also locally compact spaces are Baire spaces).

Proof: Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of dense open subsets of X.

We want to show that $\forall V \subset X$ open set, $V \cap \bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$.

Since U_0 is open dense, $\exists x_0 \in U_0$, $r_0 > 0$ so that $\overline{B(x_0, r_0)} \subset V \cap U_0$.

Since U_1 is open dense, $\exists x_1 \in U_1$, $r_1 > 0$ so that $\overline{B(x_1, r_1)} \subset B(x_0, r_0) \cap U_1$

By induction, $\exists x_n \in U_n$, $r_n > 0$ so that $\overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1}) \cap U_n$.

$\overline{B(x_n, r_n)}$ is a nested sequence of compacts in a complete metric space so $\exists x_\infty \in \bigcap \overline{B(x_n, r_n)} \subset V \cap \bigcap_{n\in\mathbb{N}} U_n$.

(Alternatively, pick r_n so that $r_n < \frac{1}{n}$ $\forall n$, and (x_n) is a Cauchy sequence $\leadsto$ converges, and the limit is in $\overline{B(x_n, r_n)}$ since it is closed. $\square$

We say that a property is true for <u>generic</u> $x \in X$ if it holds for a countable intersection of dense open subsets (sometimes called a dense G_δ-set)

(Notice that any Riemann surface is a Baire space)

Proposition: For a generic choice of $z \in J(f)$, the forward orbit $O_f(z) = \{ f^n(z) \mid n \in \mathbb{N} \}$ is dense in $J(f)$.

Proof: Consider $J(f) \subset \hat{\mathbb{C}}$ with the metric induced by the spherical metric in $\hat{\mathbb{C}}$. Since $J(f)$ is compact $\forall j \in \mathbb{N}^*$ we can cover $J(f)$ with finitely many balls $B_{j,k}$ of radius $\frac{1}{j}$ (w.r.t. the spherical metric). $(k = 1 \dots K(j))$.

For any $B_{j,k}$, the set $U_{j,k} = \bigcup_{n \in \mathbb{N}} f^{-n}(B_{j,k})$ is dense in J, i.e. $\overline{U_{j,k}} = J$. Set $U = \bigcap_{j,k} U_{j,k}$, which is a countable intersection of open dense subsets of J.

Then $\forall z \in U$ $\Rightarrow$ $\forall j,k$, $\exists n$ s.t. $f^n(z) \in B_{j,k}$, and $O_f(z)$ is dense in $J(f)$ □

Complementary properties. (Optional)

Proposition: For any open $U \subset \hat{\mathbb{C}}$ such that $J \cap U \neq \emptyset$, then $\forall n \gg 0$ we have $f^n(U) \supset J(f)$.

Proof. Since $J(f)$ has no isolated points, we can find three small balls B_1, B_2, B_3 of radius r so that $B_j \cap J(f) \neq \emptyset$, $B_j \subset U$, $\overline{B_j}$ all disjoint.

First we show that $\forall j = 1,2,3$, $\exists k = k(j) \in \{1,2,3\}$ and $n = n(j) \in \mathbb{N}$ so that $f^n(B_j) \supset B_k$.

In fact, if this is not true, $\forall n \in \mathbb{N}$, $\exists \alpha_n \in B_1$, $\beta_n \in B_2$, $\gamma_n \in B_3$, so that $f^n(B_j) \subset \hat{\mathbb{C}} \setminus \{\alpha_n, \beta_n, \gamma_n\}$

Take $\Phi_n \in \mathrm{Aut}(\hat{\mathbb{C}})$ to be the unique Möbius map so that $\Phi_n(0) = \alpha_n$, $\Phi_n(1) = \beta_n$, $\Phi_n(\infty) = \gamma_n$. Then $\Phi_n^{-1} \circ f^n$ sends U_j to $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$, and hence it forms a normal family

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Hence any sequence in $\{\Phi_n^{-1}\circ f^n\}$ admits a subsequence $\Phi_{n_k}^{-1}\circ f^{n_k}$ converging locally uniformly to some $g: U_j \to \hat{\mathbb{C}}$.

Up to taking a subsequence, we may assume that $\alpha_{n_k}\to\alpha$, $\beta_{n_k}\to\beta$, $\gamma_{n_k}\to\gamma$ (all distinct).

Then, if $\Phi\in \mathrm{Aut}(\hat{\mathbb{C}})$ is the unique Moebius map satisfying $\Phi(0)=\alpha$, $\Phi(1)=\beta$, $\Phi(\infty)=\gamma$, we have that $f^{n_k}\to\Phi\circ g$ locally uniformly.

To see this, it suffices to show that $\Phi_n\to\Phi$ uniformly on $\hat{\mathbb{C}}$. This follows from the fact that $\Phi(z)=\dfrac{\gamma(\beta-\alpha)z+\alpha(\gamma-\beta)}{(\beta-\alpha)z+\gamma-\beta}$, that we can use to get uniform estimates from the convergence $\alpha_{n_k}\to\alpha$, $\beta_{n_k}\to\beta$, $\gamma_{n_k}\to\gamma$.

The formula is obtained from the cross ratio: $\dfrac{(\Phi(z)-\alpha)(\gamma-\beta)}{(\gamma-\Phi(z))(\beta-\alpha)}=\dfrac{(z-0)(\infty-1)}{(\infty-z)(1-0)}$

It follows that $B_j\subset F(f)$, a contradiction since $B_j\cap J(f)\neq\emptyset$.

We showed that $\forall j\ \exists\, n(j), k(j)$ so that $f^n(B_j)\supset B_k$.

The map $k: \{1,2,3\}\circlearrowleft$ must have a fixed point, and we deduce that $\exists j, n$ so that $f^n(U_j)\supset U_j$.

Apply the transitivity property to $g=f^n$ and U_j:

$\bigcup_{m\geq 0} g^m(U_j)\supset J(g)=J(f)$. But $(g^m(U_j))_m$ is an increasing sequence of open sets covering the compact $J(f)$ $\Rightarrow$ $\exists\, m\in\mathbb{N}$ so that $g^m(U_j)\supset J(f)$.

Hence $f^{\overset{=n_0}{nm}}(U)\supset J(f)$. (since $U_j\subset U$)

Note that $J(f)=f(J(f))\subset f^{n_0+1}(U)$, and $f^n(U_j)\supset J(f)$ $\forall n\geq n_0$.

$\square$

4.18

Proposition: if $J(f)$ is disconnected, then $\forall z \in J(f)$ is an accumulation point of infinitely many distinct components of $J(f)$

Proof: Let $K = \{ z \in J(f) \mid z$ is an accumulation point for ∞-many components$\}$.

Since $J(f)$ has ∞-many components, and it is compact, $K \neq \emptyset$.

Moreover K is closed ($z_n \to z$, $z_n \in K \Rightarrow \forall U \ni z$, $z_n \in U$ for $n >> 0$, and U contains ∞-many components of $J(f)$ $\Rightarrow z \in K$).

If $z \in K$, then $f^{-1}(z) \subset K$: in fact $\forall w \in f^{-1}(z)$, f acts as a local diffeomorphism at w, and w is ~~[illegible]~~ accumulated by ∞-many components.

Hence $\mathcal{O}_f^-(z) \subset K$. But $\overline{\mathcal{O}_f^-(z)} = J$, hence $J \subset K$, and we are done $\square$